EFEKTYVŪS ĮRODYMO METODAI ŽINIŲ LOGIKOMS, PAGRĮSTOMS REFLEKSYVAUS BENDRO ŽINOJIMO LOGIKA

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Santrauka
The common knowledge logics are import class of non-classical logics and play a significant role in several areas of computer science, artificial intelligence, game theory, economics and etc. Common knowledge logics are based on multi-modal logics extended with common knowledge operator. Common knowledge operator satisfies some induction like axioms. In derivation this induction-like tool is realized using loop-type axioms. Determination of these loop-type axioms involves creating a new “good loop” in contrast to “bad loops” and the new loop checking along with ordinary non-induction-type loop checking.

Reikšminiai žodžiai: Refleksyvaus bendrojo žinojimo logika, žinių logikos, efektyvūs įrodymo metodai, Analitiniai skaičiavimai, Pakartotino sugrįžimo (backtracking) efektyvūs metodai, Efektyvūs įrodymų baigtinumo tikrinimo metodai, bendrojo žinojimo logika.
Turinys

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1 Įvadas

Parengtas mokslinis straipsnis „Two complete finitary sequent calculi for reflexive common knowledge“

2 Two complete finitary sequent calculi for reflexive common knowledge

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Abstract:
This paper discusses the use of complete sequent calculi for reflexive common knowledge logic. Description of language and complete infinitary calculus for RCL is presented. Then finitary calculi \( RCL_f \) and \( RCL_L \) are introduced and completeness of finitary calculi \( RCL_f \) and \( RCL_L \) is proven.

Key words: common knowledge logic, reflexive common knowledge logic, sequent calculi

1. Introduction

A reflexive common knowledge logic (RCL) containing individual knowledge operators, reflexive “common knowledge” and “everyone knows” operators is considered. Complete sequent calculi for reflexive common knowledge logic is discussed. Finitary calculi \( RCL_f \) and \( RCL_L \) are introduced and completeness of these calculi is obtained using completeness of the infinitary calculus for RCL.

2. Description of language and complete infinitary calculus for RCL.

The language of considered \( RCL \) contains a set of propositional symbols \( P, P_1, P_2, \ldots, Q, Q_1, Q_2, \ldots \) the set of logical connectives \( \supset, \land, \lor, \neg \); finite set of agent constants \( i, i_1, i_2, \ldots \); multiple knowledge modality \( K(i) \), where \( i \) is an agent constant; everyone knows operator \( E \); common knowledge operator \( C \).

A formula of \( RCL \) is defined inductively as follows: every propositional symbol is a formula; if \( A, B \) are formulas, then \( (A \supset B), (A \land B), (A \lor B), \neg(A) \) are formulas; if \( i \) is an agent, \( A \) is a formula, then \( K(i)A \) is a formula; if \( A \) is a formula, then \( E(A) \) and \( C(A) \) are formulas. The operator \( K(i) \) behaves as modality of multi-modal logic \( K_n \).
The formula $K(i)A$ means “agent $i$ knows $A$”. The formula $E(A)$ means “every agent knows $A$”, i.e. $E(A) = \wedge_{i=1}^{n} K(i)A$ ($n$ is a number of agents). The formula $C(A)$ means “$A$ is common knowledge of all agents”; it is assumed that there is perfect communication between agents. The operator $C$ and $E$ behave as modalities of modal logic $S5$. In addition these operators satisfy the following powerful properties: $C(A) = A \wedge E(C(A))$ (fixed point) and $A \wedge C(A \supset E(A)) \supset C(A)$ (induction). Formal semantics of the formulas $K(i), E(A), C(A)$ are defined as in the reflexive common knowledge logic[3].

Bellow we consider calculi based on sequents, i.e., formal expressions $A_1, ..., A_k \rightarrow B_1, ..., B_m$, where $A_1, ..., A_k$ ($B_1, ..., B_m$) is a multiset of arbitrary formulas. The infinatary calculus, denoted by $RCL_{\omega}$, for $RCL$ is defined by following postulates [3].

Axiom: $\Gamma, A \rightarrow \Delta, A$

Rules consist of logical rules and modal ones. Logical rules consist of traditional invertible rules for logical symbols.

Modal rules:

\[
\begin{align*}
\Gamma \rightarrow A \\
\Pi, K_i \Gamma \rightarrow \Delta, K_i(A) (K_i),
\end{align*}
\]

where $K_i \Gamma = K_i A_1, ..., K_i A_n (n \geq 0)$; $\Pi, \Delta$ consist of multisets of arbitrary formulas.

\[
\begin{align*}
\Lambda_{i=1}^{m} K_i(A), \Gamma \rightarrow \Delta (E \rightarrow) \\
E(A), \Gamma \rightarrow \Delta (E \rightarrow) \\
\Gamma \rightarrow \Delta, \Lambda_{i=1}^{m} K_i(A) (\rightarrow E) \\
\Gamma \rightarrow \Delta, E(A)
\end{align*}
\]

where $m$ is number of agents.

\[
\begin{align*}
A, E(C(A)), \Gamma \rightarrow \Delta (C \rightarrow) \\
C(A), \Gamma \rightarrow \Delta \\
\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, E(A); ...; \Gamma \rightarrow \Delta, E^{k}(A) ... (\rightarrow C_{\omega})
\end{align*}
\]

where $E^{0}(A) = A; E^{k}(A) = E(E^{k-1}(A)), k \geq 1$

It is known (see e.g. [3]) that calculus $RCL_{\omega}$is sound and complete.

\[
\Gamma \rightarrow A \\
\Pi, E(\Gamma) \rightarrow \Delta, E(A) (E)
\]

The rule is derivable in $RCL^*$ where $RCL^*$ is obtained from $RCL_{\omega}$by dropping the rule $(\rightarrow C_{\omega})$.

Let $\Gamma = A_1, ..., A_n$ then derivability of (E) is carried out in the following way:

\[
\begin{align*}
& A_1, ..., A_n \rightarrow A \\
& \rightarrow K_i(A_1), ..., K_i(A_n) \rightarrow \cdots \rightarrow K_i(A) ... (K_i) \\
& \Pi, E(A_1), ..., \Lambda_{i=1}^{m} K_i(A_i), E(A_n) \rightarrow \Lambda_{i=1}^{m} K_i(A) (\rightarrow \wedge) (E \rightarrow), (\rightarrow E)
\end{align*}
\]

Derivations in $RCL_{\omega}$ are built in the form of the infinite tree, each branch of this tree is finitary. The height of a derivation $D$ (denoted by $O(D)$) is evaluated in ordinals.

A derivation $D$ in $RCL_{\omega}$ is called atomic if all axioms occurring in $D$ are the form $\Gamma, P \rightarrow \Delta, P$.

Lemma 1.1. An arbitrary derivation in $RCL_{\omega}$ may be transformed into an atomic one.

Proof. Let us denote by $g(A)$ the complexity of $A$ defined by the number of occurrences of logical and knowledge operators $C, E, K_i$ in $A$. The lemma is moved by induction on $g(A)$.
It is easy to see that all rules of $RCL_\omega$, except $(K_i)$ are invertible. Let us present a specialization of the rule $(K_i)$ which is existential invertible.

A sequent $S$ is a primary if $S = \Sigma_1, K(\Gamma_1) \rightarrow \Sigma_2, K(\Gamma_2)$, where $\Sigma_i (i \in \{1,2\})$ is empty or consists of propositional symbols, $K(\Gamma_i) (i \in \{1,2\})$ is empty or consists of formulas of the shape $K_i(A)$.

Lemma 1.2. By backward applications of rules, except $(K_i)$ of $RCL_\omega$ any sequent $S$ can be reduced to a set of primary sequents $S_1, ..., S_n (n \geq 1)$ such that if $RCL_\omega \vdash S_i$ then $\forall l (l \geq 1) RCL_\omega \vdash S_i$.

Proof: follows from invertibility of rules $RCL_\omega$ except $(K_i)$.

Let $RCL'_\omega$ be the calculus obtained from $RCL_\omega$ replacing the rule $(K_i)$ by the following one:

$$\Gamma_p \rightarrow A$$

$$\Sigma_1, K_1^a(\Gamma_1), ..., K_n^a(\Gamma_n) \rightarrow \Sigma_2, K_1^s(\Delta_1), ..., K_n^s(\Delta_n), A, K_1^s(A), ..., K_m^s(\Delta_m) (K'_i)$$

$n \geq 0, m \geq 0; K_i^a = K_i^s, \Sigma_1 \cap \Sigma_2 = \emptyset$.

Lemma 1.3 (existential invertibility of the rule $(K'_i)$ in $RCL_\omega$). Let $S = \Sigma_1, K_1^a(\Gamma_1), ..., K_n^a(\Gamma_n) \rightarrow \Sigma_2, K_1^s(\Delta_1), ..., K_m^s(\Delta_m)$ be a primary sequent satisfying the condition of the conclusion of $(K'_i)$ and let $RCL_\omega \vdash \psi S$, then there exists a formula $K_i^s(A)$ such that $RCL_\omega \vdash \Gamma_p \rightarrow A$.

Proof. From Lemma 1.1 it follows that all axioms in $D$ are atomic ones. Another hand, $\Sigma_1 \cap \Sigma_2 = \emptyset$ therefore $h(D) > 1$. Therefore from the scope of the rule $(K'_i)$ it follows that there exists a formula $K_i^s(A)$ from the succedent of $S$ such that $RCL_\omega \vdash \Gamma_p \rightarrow A$.

3. Finitary calculi $RCL_1$ and $RCL_L$

Infinitary calculus $RCL_\omega$ possesses the following beautiful property: it allows to present simple and evident completeness proof (see e.g. [3]). Despite of this property: all derivation containing infinitary rule ($\rightarrow C_\omega$) are informal. To avoid this bad property several finitary complete sequent calculi for $RCL$ can be presented.

(1) Calculus containing invariant-like rule

The finitary calculus $RCL_1$ is obtained from the calculus $RCL_\omega$ replacing infinitary rule ($\rightarrow C_\omega$) by following (cut) - like rule:

$$\Gamma \rightarrow \Delta, I; I \rightarrow E(I); I \rightarrow A$$

$$\Gamma \rightarrow \Delta, C(A) \rightarrow (C_1)$$

where the formula I (called an invariant formula) is constructed from subformulas of formulas in the conclusion of the rule. There are some works in which constructive methods for finding invariant formulas in sequent calculi of epistemic logic are presented, e.g. [4], [5]. Using these methods we can find invariant formulas and for the rule ($\rightarrow C_1$)

(2) Calculus containing weak-induction like rule and loop axiom.

The finitary calculus $RCL_L$ is obtained from the calculus $RCL_1$ in the following way:

(a) replacing the invariant rule ($\rightarrow C_1$) by the following rule:

$$\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, E(C(A)) \rightarrow (C_L)$$

This rule corresponds to the so-called weak-induction axiom: $A \land E(C(A)) \Rightarrow C(A)$.

(b) adding loop-type axioms as follows: a sequent $S'$ is a loop type axiom if (1) $S'$ is above a sequent $S$ on a branch of derivation tree, (2) $S'$ is such that it subsumes $MII$-DS-09P-13-12
$S'$ ($S \geq S'$ in notation), i.e. we can get $S'$ from $S$ using structural rules of weakening and contraction, in separate case $S = S'$; (3) there is right premise of $(\rightarrow C_L)$ between $S$ and $S'$.

The completeness of finitary calculi $RCL_I$ and $RCL_L$ is obtained proving that the calculi $RCL_\omega$, $RCL_I$, and $RCL_L$ are equivalent to each other. The completeness of $RCL_\omega$ is used.

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3 Some sequent calculi for reflexive common knowledge

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Abstract:
This paper discusses the use of complete sequent calculi for reflexive common knowledge logic. Description of language and complete infinitary calculus for RCL is presented. Then finitary calculi $RCL_I$ and $RCL_L$ are introduced and completeness of finitary calculi $RCL_I$ and $RCL_L$ is proven.

Key words: reflexive common knowledge logic, sequent calculi, calculi for reflexive common knowledge logic

4. Introduction

A reflexive common knowledge logic (RCL) containing individual knowledge operators, reflexive “common knowledge” and “everyone knows” operators is considered. The use of complete sequent calculi for reflexive common knowledge logic is discussed, finitary calculi with invariant-like rule and with looping axioms are introduced and completeness of these calculi is obtained.

5. Description of language and complete infinitary calculus for RCL.

The language of considered RCL contains a set of propositional symbols $P, P_1, P_2, \ldots$, $Q, Q_1, Q_2, \ldots$ the set of logical connectives $\Rightarrow, \Lambda, \lor, \neg$; finite set of agent constants $i, i_1, i_2, \ldots$; multiple knowledge modality $K(i)$, where $i$ is an agent constant; everyone knows operator $E$; common knowledge operator $C$.

A formula of RCL is defined inductively as follows: every propositional symbol is a formula; if $A, B$ are formulas, then $(A \Rightarrow B), (A \land B), (A \lor B), \neg (A)$ are formulas; if $i$ is an agent, $A$ is a formula, then $K(i)A$ is a formula; if $A$ is a formula, then $E(A)$ and $C(A)$ are formulas. The operator $K(i)$ behaves as modality of multi-modal logic $K_n$. 

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The formula $K(i)A$ means “agent $i$ knows $A$”. The formula $E(A)$ means “every agent knows $A$”, i.e. $E(A) = \bigwedge_{i=1}^n K(i)A$ ($n$ is a number of agents). The formula $C(A)$ means “$A$ is common knowledge of all agents”; it is assumed that there is perfect communication between agents. The operator $C$ and $E$ behave as modalities of modal logic $S5$. In addition these operators satisfy the following powerful properties: $C(A) = A \land E(C(A))$ (fixed point) and $A \land C(A) \supset E(A)) \supset C(A)$ (induction). Formal semantics of the formulas $K(i), E(A), C(A)$ are defined as in the reflexive common knowledge logic[3].

Below we consider calculi based on sequents, i.e., formal expressions $A_1, ..., A_k \rightarrow B_1, ..., B_m$, where $A_1, ..., A_k (B_1, ..., B_m)$ is a multiset of arbitrary formulas. The infinitary calculus, denoted by $RCL_\omega$, for $RCL$ is defined by following postulates. (see e.g. [3]).

**Axiom:** $\Gamma, A \rightarrow \Delta, A$

Rules consist of logical rules and modal ones. Logical rules consist of traditional invertible rules for logical symbols.

**Modal rules:**

$$\frac{\Gamma \rightarrow A}{\Pi, K_i \Gamma \rightarrow \Delta, K_i(A)} (K_i),$$

where $K_i \Gamma = K_i A_1, ..., K_i A_n (n \geq 0)$; $\Pi, \Delta$ consist of multisets of arbitrary formulas.

$$\frac{\bigwedge_{i=1}^m K_i(A), \Gamma \rightarrow \Delta}{\Pi, E(A), \Gamma \rightarrow \Delta} (E \rightarrow)$$

where $m$ is number of agents.

$$\frac{A, E(C(A)), \Gamma \rightarrow \Delta}{C(A), \Gamma \rightarrow \Delta} (C \rightarrow)$$

$$\frac{\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, E(A); \ldots; \Gamma \rightarrow \Delta, E^k(A) \ldots (\rightarrow C_\omega)}{\Gamma \rightarrow \Delta, C(A)}$$

where $E^0(A) = A; E^k(A) = E \left( E^{k-1}(A) \right), k \geq 1$

It is known (see e.g. [3]) that calculus $RCL_\omega$ is sound and complete.

$$\frac{\Pi, E(\Gamma) \rightarrow \Delta, E(A)}{(E)}$$

The rule is derivable in $RCL^*$ where $RCL^*$ is obtained from $RCL_\omega$ by dropping the rule $(\rightarrow C_\omega)$.

Let $\Gamma = A_1, ..., A_n$ then derivability of $(E)$ is carried out in the following way:

$$\frac{\Pi, E(A_1), ..., E(A_n) \rightarrow E(A)}{\Pi, K_i(A_1), ..., K_i(A_n) \rightarrow \ldots \rightarrow K_i(A) \ldots (K_i) (\land \rightarrow)(\land)}$$

Derivations in $RCL_\omega$ are built in the form of the infinite tree, each branch of this tree is finitary. The height of a derivation $D$ (denoted by $O(D)$) is evaluated in ordinals.

A derivation $D$ in $RCL_\omega$ is called atomic if all axioms occurring in $D$ are the form $\Gamma, P \rightarrow \Delta, P$.

**Lemma 1.1.** An arbitrary derivation in $RCL_\omega$ may be transformed into an atomic one.

**Proof.** Let us denote by $g(A)$ the complexity of $A$ defined by the number of occurrences of logical and knowledge operators $C, E, K_i$ in $A$. The lemma is proved by induction on $g(A)$.
It is easy to see that all rules of \( RCL_\omega \), except \((K_i)\) are invertible. Let us present a specialization of the rule \((K_i)\) which is existential invertible.

A sequent \( S \) is a primary if \( S = \Sigma_1, K(\Gamma_1) \rightarrow \Sigma_2, K(\Gamma_2) \), where \( \Sigma_i (i \in \{1,2\}) \) is empty or consists of propositional symbols, \( K(\Gamma_i) (i \in \{1,2\}) \) is empty or consists of formulas of the shape \( K_i(A) \).

**Lemma 1.2.** By backward applications of rules, except \((K_i)\) of \( RCL_\omega \) any sequent \( S \) can be reduced to a set of primary sequents \( S_1, ..., S_n (n \geq 1) \) such that if \( RCL_\omega \vdash S_i \) then \( \forall l \geq 1)RCL_\omega \vdash S_i \)

**Proof:** follows from invertibility of rules \( RCL_\omega \) except \((K_i)\).

Let \( RCL'\omega \) be the calculus obtained from \( RCL_\omega \) replacing the rule \((K_i)\) by the following one:

\[
\Gamma_p \rightarrow A
\]

\[
\Sigma_1, K_1^a(\Gamma_1), ..., K_n^a(\Gamma_n) \rightarrow \Sigma_2, K_1^s(\Delta_1), ..., K_m^s(A), ..., K_m^s(\Delta_m)
\]

\( n \geq 0, m \geq 0; K_i^a = K_i^a, \Sigma_1 \cap \Sigma_2 = \emptyset, p \in \{1, ..., n\} \).

**Lemma 1.3** (existential invertability of the rule \((K_i)\) in \( RCL_\omega \)). Let \( S = \Sigma_1, K_1^a(\Gamma_1), ..., K_n^a(\Gamma_n) \rightarrow \Sigma_2, K_1^s(\Delta_1), ..., K_m^s(\Delta_m) \) be a primary sequent satisfying the condition of the conclusion of \((K_i^s)\) and let \( RCL_\omega \vdash \dotted{D} S \), then there exists a formula \( K_i^s(A) \) such that \( RCL_\omega \vdash \Gamma_p \rightarrow A \).

**Proof.** From **Lemma 1.1** it follows that all axioms in \( D \) are atomic ones. Another hand, \( \Sigma_1 \cap \Sigma_2 = \emptyset \) therefore \( h(D) > 1 \). Therefore from the scope of the rule \((K_i^s)\) it follows that there exists a formula \( K_i^s(A) \) from the succedent of \( S \) such that \( RCL_\omega \vdash \Gamma_p \rightarrow A \).

6. **Finitary calculi \( RCL_1 \) and \( RCL_1^l \)**

Infinitary calculus \( RCL_\omega \) possesses the following beautiful property: it allows to present simple and evident completeness proof (see e.g. [3]). Despite of this property: all derivation containing infinitary rule \((\rightarrow C_\omega)\) are informal. To avoid this bad property several finitary complete sequent calculi for \( RCL_\omega \) can be presented.

**3** **Calculi containing invariant-like rule**

The finitary calculus \( RCL_1 \) is obtained from the calculus \( RCL_\omega \) replacing infinitary rule \((\rightarrow C_\omega)\) by following (cut) - like rule:

\[
\Gamma \rightarrow \Delta, I; I \rightarrow E(I); I \rightarrow A
\]

\[
\Gamma \rightarrow \Delta, C(A) \rightarrow (\rightarrow C_1)
\]

where the formula \( I \) (called an invariant formula) is constructed from subformulas of formulas in the conclusion of the rule. There are some works in which constructive methods for finding invariant formulas in sequent calculi of temporal and dynamic logics are presented, e.g. [5]. Using these methods we can find invariant formulas and for the rule \((\rightarrow C_1)\)

**Example 2:** Let \( S = P, C(P \supset E(P)) \rightarrow C(P) \)

i.e. the same sequent as in Example 1. Let us construct the derivation of \( S \) in \( RCL_1 \). As invariant formula we take the formula \( P \wedge C(P \supset E(P)) \):

\[
P \rightarrow P; P, C(P \supset E(P)) \rightarrow C(P \supset E(P))
\]

\[
P, C(P \supset E(P)) \rightarrow P \wedge C(P \supset E(P)); S^* ; P \rightarrow P \rightarrow (\rightarrow \lambda)
\]

\[
P, C(P \supset E(P)) \rightarrow C(P)
\]

To conclude derivation it is necessary to construct a derivation \( D^* \) of the sequent \( S^* = P \wedge C(P \supset E(P)) \rightarrow E(P \wedge C(P \supset E(P))) \):

\[
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\]

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In the presented derivation $D^*$ the rule $(E)$ is used and the derivability of $(E)$ in $RCL^*$ was presented in Example 1.

Sequent calculi for temporal and dynamic logics were explained by many authors (see, e.g. [4], [5], [6]).

(4) Calculus containing weak-induction like rule and loop axiom.

The finitary calculus $RCL_L$ is obtained from the calculus $RCL_I$ in the following way:
(a) replacing the invariant rule $(\rightarrow C_I)$ by the following rule:
$$
\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, E(C(A))
$$
$$
\Gamma \rightarrow \Delta, C(A) \quad (\rightarrow C_L)
$$
This rule corresponds to the so-called weak-induction axiom: $A \land E(C(A)) \Rightarrow C(A)$.
(b) adding loop-type axioms as follows: a sequent $S'$ is a loop type axiom if (1) $S'$ is above a sequent $S$ on a branch of derivation tree, (2) $S'$ is such that it subsumes $S' (S \supseteq S' \text{ in notation})$, i.e. we can get $S'$ from $S$ using structural rules of weakening and contraction, in separate case $S = S'$; (3) there is right premise of $(\rightarrow C_L)$ between $S$ and $S'$.

Example 3.
Let $S = P, C(A) \rightarrow C(P)$, where $A = (P \Rightarrow E(P))$. The derivation of $S$ in $RCL_L$ is as follows:
$$
P \rightarrow P \rightarrow \ldots; P, E(P), E(C(P \Rightarrow E(P))) \rightarrow E(C(P)) \quad (E)
$$
$$
P \rightarrow P \rightarrow \ldots; P, C(P \Rightarrow E(P)) \rightarrow E(C(P)) \rightarrow C_L
$$
$$
S = P, C(A) \rightarrow C(P) \quad (\rightarrow C_L)
$$
It is obvious that $S = S'$ and loop-type conditions are satisfied for $S'$, therefore $RCL_L \vdash S$.

Sequent calculi for BDI logic was considered (see [5])

7. Completeness of finitary calculi $RCL_I$ and $RCL_L$

First we prove
Lemma 3.1. For each sequent $S: RCL_L \vdash S \Rightarrow RCL_I \vdash S$
Firstly each looping axiom is replaced by the application of the rule $(\rightarrow \square_i)$. Now let us eliminate the application of the rule $(\rightarrow \square_i)$.
Proof. Let $S_1 = \Gamma \rightarrow \Delta, A$ and $S_2 = \Gamma \rightarrow \Delta, E(C(A))$ be the left and the right premises of the rule $(\rightarrow C_L)$ . It is easy to verify that $RCL_I \vdash S_2^* = A \land E(C(A)) \rightarrow E(A \land E(C(A)))$ i.e.; the middle premise of $(\rightarrow C_I)$. Applying $(\land \Rightarrow)$ from $S_1, S_2$ we get $RCL_I \vdash S_1^* = \Gamma \rightarrow \Delta, A \land E(C(A))$ . It is obvious that $RCL_I \vdash S^* = A \land E(C(A)) \rightarrow A$. Now applying to $S_1^*, S_2^*, S_1$ the rule $(\rightarrow C_I)$ we get that $RCL_I \vdash \Gamma \rightarrow \Delta, C(A)$, i.e. the conclusion of $(\rightarrow C_I)$.

Lemma 3.2. The rule.
\[
\frac{S_1 = \Gamma \rightarrow \Delta, A; S_2 = A, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda} \quad (\text{cut})
\]

is admissible in \( RCL_L \).

Proof. The proof is carried out in traditional way using double induction \((|A|, h(S_1) + h(S_2))\), where \(|A|\) is complexity of \( A \); \( h(S_i) \) \((i \in \{1,2\})\) is the height of left and right premise of \((\text{cut})\). During the proof the invertibility of rules is used. Let us consider only the case when \( S_1 \) and \( S_2 \) are looping axioms and \( A \) is propositional symbol. In this case the conclusion of \((\text{cut})\) is also looping axiom.

To prove that \( RCL_L \vdash S \Rightarrow RCL_L \vdash S \) let us introduce a semi-Hilbert type calculus \( HRCL \) which is obtained from the calculus \( RCL_L \) replacing the invariant rule \((\rightarrow C_i)\) with the induction axiom \( A, C(A \supset E(A)) \rightarrow C(A) \) \((\ast)\) and adding the \((\text{cut})\) rule. Completeness of \( HRCL \) is proved almost in the same way as completeness of Hilbert type version of \( RCL_L \) (see [3]).

From completeness of \( HRCL_L \) and \( RCL_\omega \) we get that the calculi \( HRCL_L \) and \( RCL_\omega \) are equivalent.

Lemma 3.3. \( HRCL_L \vdash S \Rightarrow RCL_L \vdash S \)

Proof. To prove the lemma is sufficient to construct derivation of induction axiom \((\ast)\) in \( RCL_L \):

\[
\frac{S' = A, C(A \supset E(A)) \rightarrow C(A) \quad \ldots A \rightarrow A \ldots; A, E(A), E(C(A \supset E(A)) \rightarrow E(C(A)))}{S^* = A, C(A \supset E(A)) \rightarrow C(A)} \quad (\text{cut}), (\supset \rightarrow), (\rightarrow C_L)
\]

Since \( S^* = S' \), \( S' \) is looping axiom, therefore \( RCL_L \vdash S \)

Lemma 3.4. \( RCL_L \vdash S \Rightarrow RCL_L \vdash S \)

Proof. From the given derivation of \( S \) and from completeness of \( HRCL_L \), and Lemma 3.3. we get that \( RCL_L \vdash S \).

Theorem 3.1. The calculi \( RCL_\omega, RCL_L \) and \( RCL_L \) are equivalent.

Proof: follows from Lemmas 3.6, 3.1, 3.4 and Conclusion from Lemma 3.5.

Theorem 3.2. The calculi \( RCL_L \) and \( RCL_L \) are sound and complete.

Proof: follows from completeness and soundness of the calculus \( RCL_\omega \) and Theorem 3.1.

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